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DIMENSION THEORY AND SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

BY

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ABSTRACT

The main result of this paper is the following: If X is a compact two dimensional metric space, and $\{\varphi_i\}_{i=1}^4$ are four functions in $C(X)$, then there exists a function f in $C(X)$ which cannot be represented in the form:

$$f(x) = \sum_{i=1}^{4} g_i(\varphi_i(x)),$$

with

$$g_i \in C(R).$$

1. Introduction

The present paper deals with some questions related to the following general theorem concerning superpositions of real valued functions due to Ostrand [7]. (*All topological spaces throughout this paper are compact metric.* By $C(X)$ we denote the Banach space of real valued continuous functions on X, with the supremum norm.)

THEOREM 1. *Let* $X = X_1 \times X_2 \times \cdots \times X_k$, *with* $\dim X_i = n_i (i = 1, 2, \cdots k)$ *and* $n = \sum_{i=1}^{k} n_i$. *Then there exist functions* $\varphi_1^i, \varphi_2^i, \cdots \varphi_{2n+1}^i$ *in* $C(X_i)$, *such that to every* $f \in C(X)$ *there corresponds* $g_j \in C(R)$ $j = 1, 2, \cdots 2n+1$ *so that the representation* $f(x_1, x_2, \cdots x_k) = \sum_{j=1}^{2n+1} g_j[\varphi_j^1(x_1) + \varphi_j^2(x_2) + \cdots + \varphi_j^k(x_k)]$ *holds.*

This theorem is a generalization of the well-known theorem of Kolmogorov [5], where $X_i = [0, 1]$ for all i. There are several natural problems which arise in connection with Theorem 1. (See [9] and [10] for a survey of related questions.) We shall consider here the problem of the number of summands needed for a representation of the form given by Theorem 1. For convenience, we restate Th. 1 in the case $k = 1$.

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THEOREM 1*. *Let X be an n-dimensional space. Then there exist* $\{\varphi_i\}_{i=1}^{2n+1}$ *in* $C(X)$, *such that for every f in* $C(X)$, *there corresponds* $\{g_i\}_{i=1}^{2n+1}$ *in* $C(R)$ *so that the representation*

$$f(x) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x))$$

holds.

In Sec. 2 of this paper, we prove that in the case $n = 2$, the number $2n + 1$ is the minimal one for a representation of the type appearing in Theorem 1* (and thus in Theorem 1). More precisely, we prove:

THEOREM 2. *Let X be a 2-dimensional space, and let* $\{\varphi_i\}_{i=1}^{4}$ *be functions in* $C(X)$. *Then there exists an* $f \in C(X)$ *such that f cannot be represented in the form*

$$f(x) = \sum_{i=1}^{4} g_i(\varphi_i(x))$$

with g_i *in* $C(R)$.

In the case $n = 1$, the situation is different, and we shall consider it briefly in Sec. 3. For $n > 2$ the situation is similar to the case $n = 2$. Some of the lemmas presented here can be immediately generalized to the case $n > 2$. In others, the situation is more difficult. We shall consider the case $n > 2$ in a subsequent paper. (See note at end of paper.)

Theorem 2 is an improvement of a theorem of Bassalygo [2], who proved that in the case $X = [0, 1]^2$, three functions $\{\varphi_i\}_{i=1}^{3}$ are not enough for a representation of the type appearing in Theorem 1*. Our Theorem 2 also improves a theorem of Doss [3] who proved that in the case $n = 2$, four functions are not enough for the representation in Kolmogorov's theorem [5], provided the functions φ_j^i are monotone. We use here some of his methods.

Let us remark that the problem of the number of needed summands leads, even in the simplest case, (i.e. in Kolmogorov's theorem), to dimension theoretic problems. Therefore, the natural approach is to begin at the outset with an arbitrary n-dimensional space. (Two dimensional space in the present paper.)

2. Proof of Theorem 2

The proof of the theorem is based on a series of lemmas. The first lemma shows that two functions are not enough in Theorem 1* if $\dim X = 2$. Later, we shall show that three functions do not suffice and eventually that even four are not enough (this is the statement of Theorem 2).

LEMMA 1. *Let X be a two dimensional space, and let φ_1, φ_2 be functions in $C(X)$. Then there exists an $f \in C(X)$ such that f cannot be represented in the form:*

$$f(x) = g_1(\varphi_1(x)) + g_2(\varphi_2(x))$$

with g_i $(i = 1,2)$ real functions.

PROOF. Suppose that such an f does not exist. Then clearly, the functions φ_1, φ_2 separate the points of X, and thus the mapping $\psi: X \to R^2$, defined by $\psi(x) = (\varphi_1(x), \varphi_2(x))$, is a homeomorphism. Hence $\psi[X]$ is a two dimensional set in R^2, and therefore has a non-empty interior in R^2 ([4] p. 44). In particular, $\psi[X]$ contains a rectangle with vertices

$$(\alpha_1, \beta_1), \quad (\alpha_2, \beta_2), \quad (\alpha_1, \beta_2), \quad (\alpha_2, \beta_1).$$

Set $x_1 = \psi^{-1}(\alpha_1, \beta_1)$, $x_2 = \psi^{-1}(\alpha_2, \beta_2)$, $x_3 = \psi^{-1}(\alpha_1, \beta_2)$, $x_4 = \psi^{-1}(\alpha_2, \beta_1)$.

Let f be a function in $C(X)$ so that $f(x_1) = f(x_2) = 1$, and $f(x_3) = f(x_4) = 0$. Since f can be represented by $f(x) = g_1(\varphi_1(x)) + g_2(\varphi_2(x))$, we get

$$2 = f(x_1) + f(x_2) = g_1(\alpha_1) + g_2(\beta_1) + g_1(\alpha_2) + g_2(\beta_2) = f(x_3) + f(x_4) = 0$$

which is a contradiction.

Let us recall now some definitions and results from dimension theory. An n-dimensional (compact metric) space is called an n-dimensional Cantor manifold if it cannot be disconnected by a subset of dimension $\leqq n-2$. Clearly, an n-dimensional Cantor manifold is connected, and has dimension n at each of its points.

It is well known that every n-dimensional space contains an n-dimensional Cantor manifold ([4], pp. 93–95). Thus there will be no loss of generality if we assume in the proof of Th. 2 that X is a two-dimensional Cantor manifold. Indeed, if $Y \subset X$, and $f \in C(Y)$ cannot be represented as $f(x) = \sum_{i=1}^{4} g_i(\varphi_i(x))$ $(x \in Y)$, then such a representation is impossible for an extension $\hat{f}$ of f over X. From now on *we assume that X is a two dimensional Cantor manifold.*

A function $\psi: X \to Y$ is called zero-dimensional if $\dim \psi^{-1}(y) \leqq 0$ for every $y \in Y$. A theorem of Hurewicz ([4], p. 91) states that $\dim X \leqq \dim \psi[X]$ whenever ψ is zero-dimensional.

LEMMA 2. *Let X be a two-dimensional Cantor manifold, and let $\{\varphi_i\}_{i=1}^{k+1}$ be $k+1$ functions in $C(X)$ such that each $f \in C(X)$ admits a representation of the form $f(x) = \Sigma_{i=1}^{k+1} g_i(\varphi_i))$, with g_i in $C(R)$. Assume also that for every two dimensional space Y, and k functions $\{\psi_i\}_{i=1}^{k}$ in $C(Y)$, there exists an $f \in C(Y)$ such that f cannot be represented as:*

$$f(x) = \sum_{i=1}^{k} g_i(\psi_i(x)) \text{ with } g_i \text{ in } C(R)$$

(i.e. $k+1$ is the minimal number for two-dimensional spaces). Then every k-tuple of different functions out of the $k+1$ functions $\{\varphi_i\}_{i=1}^{k+1}$ defines a zero-dimensional mapping from X into R^k.

PROOF. It is clearly enough to consider $\psi = (\varphi_1, \varphi_2, \cdots \varphi_k)$. Suppose that $\dim \psi > 0$, i.e. that there exists an $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_k)$ in R^k such that $\dim \psi^{-1}(\alpha) \geqq 1$. Let $F = \psi^{-1}(\alpha)$. Then for $1 \leqq i \leqq k$, we have $\varphi_i[F] = \alpha_i$. Since $\{\varphi_i\}_{i=1}^{k+1}$ separate the points of X, and φ_i $1 \leqq i \leqq k$ are constant on F, φ_{k+1} must be one-to-one on F. Thus p_{k+1}/F is a homeomorphism, and $\varphi_{k+1}[F]$ is a one--dimensional set in R. Hence, $\varphi_{k+1}[F]$ contains an interval J of positive length.

Choose $x \in F$ so that $\varphi_{k+1}(x)$ is an interior point of J. By the continuity of φ_{k+1}, there exists a neighborhood U of x in X so that $\varphi_{k+1}[U] \subset J$.

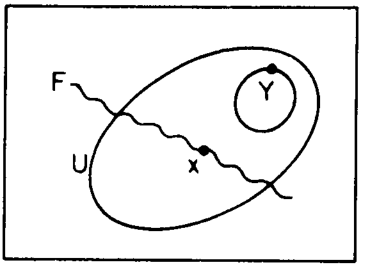

Fig. 1

F is closed and one-dimensional, thus, F is not dense in any open set of X. (Because X is a two-dimensional Cantor manifold). Hence, we can find a closed subset Y of U, with non-empty interior so that $Y \cap F = \phi$. Since Y has non-empty interior in X we have $\dim Y = 2$.

We shall show that every $f \in C(Y)$ can be represented as $f(x) = \Sigma_{i=1}^{k} g_i(\varphi_i(x))$, which will contradict the assumption of the lemma. Let f be any function in $C(Y)$, and let $\hat{f} \in C(X)$ satisfy $\hat{f}/Y = f$, and $\hat{f}/F = 0$. By the assumption of the lemma $\hat{f}$ can be represented as $\hat{f}(x)=\sum_{i=1}^{k+1} g_i(\varphi_i(x))$, $x \in X$. Without loss of generality we may assume that $g_i(\alpha_i)=0$ for $1 \leqq i \leqq k$, where α_i are the coordinates of α. We recall that for $x \in F$ we have $\varphi_i(x)=\alpha_i$ $1 \leqq i \leqq k$ and $\hat{f}(x)=0$. So that for $x \in F$ we get:

$$0=\hat{f}(x)=\sum_{i=1}^{k+1} g_i(\varphi_i(x))=\sum_{i=1}^{k} g_i(\alpha_i)+g_{k+1}(\varphi_{k+1}(x))$$

$$=g_{k+1}(\varphi_{k+1}(x)),$$

i.e. g_{k+1} vanishes on $\varphi_{k+1}[F]$, and in particular on $\varphi_{k+1}[Y]$ which is contained in $\varphi_{k+1}[F]$.

Consequently for $x \in Y$ we have:

$$f(x)=\hat{f}(x)=\sum_{i=1}^{k} g_i(\varphi_i(x)). \quad \square$$

LEMMA 3. *Let X be a space, and let $\{\varphi_i\}_{i=1}^k$ be functions in $C(X)$ such that every $f \in C(X)$ admits a representation $f(x)=\sum_{i=1}^k g_i(\varphi_i(x))$, with $g_i \in C(A_i)$ (A_i is the range of φ_i). Then there exists an $M>0$ so that each $f \in C(X)$ admits the above representation with g_i satisfying $\|g_i\| \leqq M\|f\|$. ($1 \leqq i \leqq k$).*

PROOF. Set $C=C(A_1)\times C(A_2)\times\cdots\times C(A_k)$, with the norm $\|g_i,\cdots,g_k\|_C = \max_{1\leq i\leq k}\|g_i\|_{A_i}$. Consider the operator $T: C \to C(X)$ defined by:

$$T(g_1\cdots g_k)(x)=\sum_{i=1}^{k} g_i(\varphi_i(x)).$$

Obviously T is linear and bounded. ($\|T\| \leqq k$) and by our assumption, T maps C onto $C(X)$. Thus by the open mapping theorem, T maps the unit ball B of C onto a neighborhood of the origin of $C(X)$, and therefore, there exists an $M>0$ so that $T(MB)$ contains the unit ball of $C(X)$. $\square$

LEMMA 4. *Let X be a two-dimensional Cantor manifold, let $\{\varphi_i\}_{i=1}^3$ be functions in $C(X)$ such that each $f \in C(X)$ admits a representation $f(x)=\sum_{i=1}^3 g_i(\varphi_i(x))$ with $g_i \in C(R)$, and let M be a positive number. Then there exists an $f \in C(X)$, $\|f\|=1$, such that in every representation of f in the above-mentioned type, there exists an index, i, so that $\|g_i\| \geqq M$.*

REMARK. Clearly Lemmas 3 and 4 imply that for every two-dimensional space X, and every three functions $\varphi_1, \varphi_2, \varphi_3$ in $C(X)$, there exists an $f \in C(X)$ which cannot be represented as $f(x)=\sum_{i=1}^3 g_i(\varphi_i(x))$.

PROOF OF LEMMA 4. We prove the lemma in two steps. We show first that if X contains a sequence of a certain type, the conclusion of Lemma 4 holds, and then that such a sequence does indeed exist in X. We use the following notation: the vector valued function $(\varphi_1, \varphi_2, \varphi_3,) : X \to R^3$ will be denoted by φ. The images of the elements of X under $\varphi_1, \varphi_2, \varphi_3$ will be denoted by α, β, γ respectively.

A sequence $\{x_j\}_{j=1}^k$ of elements of X will be called an alternating sequence if the following holds:

(1) $$\varphi_2(x_{2j-1}) = \varphi_2(x_{2j}) \qquad \text{for} \qquad j = 1, 2, \cdots \left[\frac{k}{2}\right]$$

(2) $$\varphi_3(x_{2j}) = \varphi_3(x_{2j+1}) \qquad \text{for} \qquad j = 1, 2, \cdots \left[\frac{k}{2}\right] - 1.$$

Two sequences $\{x_j\}_{j=1}^k$ and $\{y_j\}_{j=1}^k$ of elements of X will be called a double alternating sequence, if both $\{x_j\}_{j=1}^k$ and $\{y_j\}_{j=1}^k$ are alternating sequences, and in addition $\varphi_1(x^j) = \varphi_1(y^j)$ for $1 \leqq j \leqq k$. Table I is an example of a double alternating sequence, for $k = 6$.

TABLE I

$\varphi(x_1) = (\alpha_1, \beta_1, \gamma_1)$	$\varphi(y_1) = (\alpha_1, \tilde{\beta}_1, \tilde{\gamma}_1)$
$\varphi(x_2) = (\alpha_2, \beta_1, \gamma_2)$	$\varphi(y_2) = (\alpha_2, \tilde{\beta}_1, \tilde{\gamma}_2)$
$\varphi(x_3) = (\alpha_3, \beta_2, \gamma_2)$	$\varphi(y_3) = (\alpha_3, \tilde{\beta}_2, \tilde{\gamma}_2)$
$\varphi(x_4) = (\alpha_4, \beta_2, \gamma_3)$	$\varphi(y_4) = (\alpha_4, \tilde{\beta}_2, \tilde{\gamma}_3)$
$\varphi(x_5) = (\alpha_5, \beta_3, \gamma_3)$	$\varphi(y_5) = (\alpha_5, \tilde{\beta}_3, \tilde{\gamma}_3)$
$\varphi(x_6) = (\alpha_6, \beta_3, \gamma_4)$	$\varphi(y_6) = (\alpha_6, \tilde{\beta}_3, \tilde{\gamma}_4)$

We claim that if for every intger k, there is in X a double alternating sequence of length k, with distinct elements, then Lemma 4 is true. Indeed, let $M > 0$ be given. Let k be an even integer, $k \geqq 2M$, and let $\{x_j\}_{j=1}^k$ $\{y_j\}_{j=1}^k$ be a double alternating sequence with distinct elements. Let $L = \{x_1, x_3, \cdots x_{k-1}, y_2, y_4, \cdots, y_k\}$ and $N = \{x_2, x_4, \cdots x_k, y_1, y_3, \cdots, y_{k-1}\}$. Since all the points are distinct $L \cap N = \phi$, and hence there is an $f \in C(X)$ with $\|f\| = 1$, so that f is equal to 1 on L and to -1 on N. We shall show that this f has the desired property. Suppose f admits the representation:

$$f(x)=\sum_{i=1}^{3} g_i(\varphi_i(x)) \qquad x\in X.$$

Let $W=\Sigma_{x\in L}f(x)-\Sigma_{x\in N}f(x)$. Since $f(L)=1$ and $f(N)=-1$ and both L and N contain k points, we get that $W=2k$.

On the other hand, since $\{x_j\}_{j=1}^k\ \{y_j\}_{j=1}^k$ is a double alternating sequence we get $(m=k/2)$[†]:

$$\begin{aligned}
W&=\sum_{x\in L} f(x)-\sum_{x\in N} f(x)=\sum_{j=1}^{m} f(x_{2j-1})+\sum_{j=1}^{m} f(y_{2j})\\
&\quad-\sum_{j=1}^{m} f(x_{2j})-\sum_{j=1}^{m} f(y_{2j-1})\\
&=\sum_{j=1}^{m}\sum_{i=1}^{3} g_i(\varphi_i(x_{2j-1}))+\sum_{j=1}^{m}\sum_{i=1}^{3} g_i(\varphi_i(y_{2j}))\\
&\quad-\sum_{j=1}^{m}\sum_{i=1}^{3} g_i(\varphi_i(x_{2j}))-\sum_{j=1}^{m}\sum_{i=1}^{3} g_i(\varphi_i(y_{2j-1}))\\
&=g_3(\varphi_3(x_1))-g_3(\varphi_3(y_1))-g_3(\varphi_3(x_k))+g_3(\varphi_3(y_k))\\
&\quad+\sum_{j=1}^{k}[g_1(\varphi_1(x_j))-g_1(\varphi_1(y_j))]+\sum_{j=1}^{m}[g_2(\varphi_2(x_{2j-1}))\\
&\quad-g_2(\varphi_2(x_{2j}))]+\sum_{j=1}^{m}[g_2(\varphi_2(y_{2j}))-g_2(\varphi_2(y_{2j-1}))]\\
&\quad+\sum_{j=1}^{m}[g_3(\varphi_3(x_{2j}))-g_3(\varphi_3(x_{2j+1}))]+\sum_{j=1}^{m}[(g_3(\varphi_3(y_{2j+1}))\\
&\quad-g_3(\varphi_3(y_{2j}))]=g_3(\gamma_1)-g_3(\tilde\gamma_1)-g_3(\gamma_{m+1})+g_3(\tilde\gamma_{m+1}),
\end{aligned}$$

(where $\gamma_1=\varphi_3(x_1)$, $\tilde\gamma_1=\varphi_3(y_1)$, $\gamma_{m+1}=\varphi_3(x_k)$, $\tilde\gamma_{m+1}=\varphi_3(y_k)$).
Hence:

$$g_3(\gamma_1)-g_3(\tilde\gamma_1)-g_3(\gamma_{m+1})+g_3(\tilde\gamma_{m+1})=W=2k\geqq 4M,$$

and thus $\|g_3\|\geqq M$. It remains to show that under the assumptions of Lemma 4, there exists in X a double-alternating sequence of length k, for every even k.

[†] It might be helpful to follow this computation with the example in Table I in mind, and noting that if $\varphi(x)=(\alpha,\beta,\gamma)$ then $f(x)=g_1(\alpha)+g_2(\beta)+g_3(\gamma)$.

Set $(\varphi_1, \varphi_2) = \varphi_{1,2} : X \to R^2$, and $(\varphi_1, \varphi_3) = \varphi_{1,3} : X \to R^2$. Let us recall that by Lemmas 1 and 2, the functions $\varphi_{1,2}$, and $\varphi_{1,3}$ are zero-dimensional, and thus the image of any open set of X under each of these two functions is two--dimensional in R^2.

We introduce a special notation: Let F be a closed subset of X with a non-empty interior. We say that $F \supset_{1,2} H$ if there exists a rectangle S in R^2 with sides parallel to the axes and with positive area so that $\varphi_{1,2}(F) \supset S$ and $H = F \cap \varphi_{1,2}^{-1}(S)$. With this definition it is clear that whenever $F \supset_{1,2} H$ we have that $F \supset H$, H is closed with non-empty interior, and $\varphi_{1,2}[H]$ is a rectangle (namely the rectangle S appearing in the definition). The notation $F \supset_{1,3} H$ is defined similarly, (with $\varphi_{1,3}$ replacing $\varphi_{1,2}$). Observe that since $\varphi_{1,2}$ and $\varphi_{1,3}$ are zero-dimensional, it is possible to find for every closed subset F of X, with a non-empty interior, sets H and L so that $F \supset_{1,2} H$ and $F \supset_{1,3} L$.

Let k be any positive integer. Set $X = F_k$. Clearly $\varphi_{1,2}[F_k]$ contains a rectangle $S = \{(\alpha\beta : \alpha' \leqq \alpha \leqq \alpha'', \ \beta' \leqq \beta \leqq \beta''\}$. Let β^*, β^{**} be real so that $\beta' < \beta^* < \beta^{**} < \beta''$ (See Fig. 2). Set

$$S_k' = \{(\alpha\beta) : \alpha' \leqq \alpha \leqq \alpha''; \quad \beta' \leqq \beta \leqq \beta^*\};$$

$$S_k'' = \{(\alpha\beta) : \alpha' \leqq \alpha . \leqq \alpha''; \quad \beta^{**} \leqq \beta \leqq \beta''\}; \quad \text{and}$$

$$F_k' = F_k \cap \varphi_{1,2}^{-1}(S_k'); \quad F_k'' = F_k \cap \varphi_{1,2}^{-1}(S_k'').$$

Clearly $F_k' \cap F_k'' = \phi$ and $F_k \supset_{1,2} F_k'$; $F_k \supset_{1,2} F_k''$.

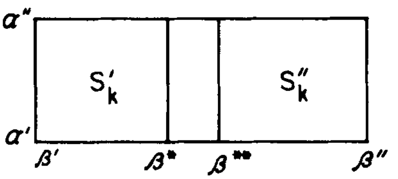

Fig. 2

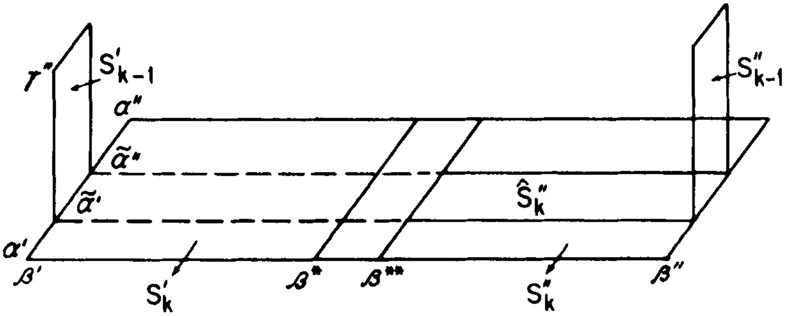

Fig. 3

The set $\varphi_{1,3}[F'_k]$ contains a rectangle S'_{k-1} in the $\alpha\gamma$ plane, of the form $S'_{k-1}=\{(\alpha,\gamma):\tilde{\alpha}'\leqq\alpha\leqq\tilde{\alpha}'';\ \gamma'\leqq\gamma\leqq\gamma''\}$. Obviously, $\alpha'\leqq\tilde{\alpha}'<\tilde{\alpha}''\leqq\alpha''$ i.e., the interval $[\alpha',\alpha'']$ contains the interval $[\tilde{\alpha}',\tilde{\alpha}'']$.

Set $F'_{k-1}=F'_k\cap\varphi_{1,3}^{-1}(S'_{k-1})$ and we have: $F_k\supset_{1,2}F'_k\supset_{1,3}F'_{k-1}$. Let us return to S''_k. It contains the rectangle $\hat{S}''_k$. $\hat{S}''_k=\{(\alpha\beta):\tilde{\alpha}'\leqq\alpha\leqq\tilde{\alpha}'',\ \beta^{**}\leqq\beta\leqq\beta''\}$ (see Fig. 3). Set $\hat{F}''_k=F''_k\cap\varphi_{1,2}^{-1}(\hat{S}''_k)$. Then we have $\varphi_1[\hat{F}''_k]=[\tilde{\alpha}',\tilde{\alpha}'']$. Now $\varphi_{1,3}[\hat{F}''_k]$ contains a rectangle S''_{k-1}, and it follows from the construction that its α side is contained in the interval $[\tilde{\alpha}',\tilde{\alpha}'']$ (see Fig. 3). Set $F''_{k-1}=\varphi_{1,3}^{-1}(S''_{k-1})\cap F''_k$. We return to S'_{k-1}. It contains a rectangle $\hat{S}'_{k-1}$, whose γ coordinates are the same as those of S'_{k-1}, and with α coordinates which are contained in the interval of α coordinates of S''_{k-1}. Put $\hat{F}'_{k-1}=\varphi_{1,3}^{-1}(\hat{S}'_{k-1})$. The set $\varphi_{1,2}[\hat{F}'_{k-1}]$ contains a rectangle S'_{k-2}, whose α coordinates are necessarily contained in the interval of α coordinates of S''_{k-1}. Set $F'_{k-2}=\varphi_{1,2}^{-1}(S'_{k-1})\cap F'_{k-1}$, and we have $F'_{k-1}\supset_{1,2}F'_{k-2}$. Continuing in this manner we construct sets $\{F'_i\}_{i=1}^k$ and $\{F''_i\}_{i=1}^k$ so that:

$$X=F_k\ \begin{cases}\underset{1,2}{\supset}F'_k\underset{1,3}{\supset}F'_{k-1}\underset{1,2}{\supset}F'_{k-2}\underset{1,3}{\supset}---\supset F'_3\underset{1,3}{\supset}F'_2\underset{1,2}{\supset}F'_1.\\[2ex] \underset{1,2}{\supset}F''_k\underset{1,3}{\supset}F''_{k-1}\underset{1,2}{\supset}F''_{k-2}\underset{1,3}{\supset}---\supset F''_3\underset{1,3}{\supset}F''_2\underset{1,2}{\supset}F''_1,\end{cases}$$

and

$$\varphi_1[F'_k]=\varphi_1[F''_k]=[\alpha',\alpha'']\supset\varphi_1[F'_{k-1}]\supset\varphi_1[F''_{k-1}]\supset\varphi_1[F'_{k-2}]$$
$$\supset\varphi_1[F''_{k-2}]\supset---\supset\varphi_1[F'_2]\supset\varphi_1[F''_1]\supset\varphi_1[F'_1]=[\alpha^*,\alpha^{**}].$$

The interval $[\alpha^*,\alpha^{**}]=\varphi_1[F'_1]$ is thus contained in the interval of α coordinates of all the rectangles appearing above.

Choose now k distinct points $\{\alpha_i\}_{i=1}^k$ in the interval $[\alpha^*,\alpha^{**}]$, and let $\{\hat{x}_i\}_{i=1}^k$ be points in F'_1 so that $\varphi_1(\hat{x}_i)=\alpha_i$. Set $\varphi_2(\hat{x}_i)=\hat{\beta}_i$ and $\varphi_3(\hat{x}_i)=\hat{\gamma}_i$. Set $x_1=\hat{x}_1$, $\beta_1=\hat{\beta}_1$, and $\gamma_1=\hat{\gamma}_1$. Since $F'_2\supset_{1,2}F'_1$, $\varphi_{1,2}[F'_1]$ is a rectangle. Since $(\alpha_1,\beta_1)=\varphi_{1,2}(x_1)$ and $(\alpha_2,\hat{\beta}_2)=\varphi_{1,2}(\hat{x}_2)$ are both in this rectangle, the point (α_2,β_1) also belongs to this rectangle. Hence there is an $x_2\in F'_1$ such that $\varphi_{1,2}(x_2)=(\alpha_2,\beta_1)$. Put $\varphi_3(x_2)=\gamma_2$, and then $\varphi(x_1)=(\alpha_1,\beta_1,\gamma_1)$ and $\varphi(x_2)=(\alpha_2,\beta_1,\gamma_2)$. Since $\hat{x}_3,x_2$ are in F'_2, and $F'_3\supset_{1,3}F'_2$ the same argument shows that there exists $x_3\in F'_2$ such that $\varphi_{1,3}(x_3)=(\alpha_3,\gamma_2)$. Put $\varphi_2(x_3)=\beta_2$ and then $\varphi(x_3)=(\alpha_3,\beta_2,\gamma_2)$. The three points we constructed this far satisfy

$$\varphi(x_1) = (\alpha_1, \beta_1, \gamma_1)$$

$$\varphi(x_2) = (\alpha_2, \beta_1, \gamma_2)$$

$$\varphi(x_3) = (\alpha_3, \beta_2, \gamma_2)$$

which are precisely the first three rows of the left column of Table I. By an obvious induction we continue the construction of the x_i $i = 1, 2 -- k$ so that $\{x_i\}_{i=1}^k$ will be an alternating sequence. The $\{y_i\}_{i=1}^k$ are constructed in the same manner, using the sets F_i'', one has only to choose the points $\{\hat{y}_i\}_{i=1}^k$ so that $\varphi_1(\hat{y}_i) = \alpha_i$ for all i. This is possible since $\varphi_1[F_1''] \supset [\alpha^*, \alpha^{**}] \supset \{\alpha_i\}_{i=1}^k$. It is easy to see that $\{x_i\}_{i=1}^k \cap \{y_i\}_{i=1}^k = \phi$ since $\{x_i\}_{i=1}^k \subset F_k'$ and $\{y_i\}_{i=1}^k \subset F_k''$ and $F_k' \cap F_k'' = \phi$. (The $\{x_i\}_{i=1}^k$ and similarly the $\{y_i\}_{i=1}^k$ are mutually distinct since $\alpha_i \neq \alpha_j$ for $i \neq j$). This proves Lemma 4. □

The next Lemma is of a general geometric nature.

LEMMA 5. *Let W be a two dimensional Cantor manifold contained in R^3. If W is not contained in a plane perpendicular to one of the axes of R^3, then at least two of its three projections on the two-dimensional planes determined by the axes are two dimensional.*

PROOF. We denote the axes of R^3 by X, Y, Z, the projections on them by P_x, P_y, P_z, and the projections on the axis planes by P_{xy}, P_{xz}, P_{yz}.

We shall prove that if $\dim P_{xy}[W] = \dim P_{xz}[W] = 1$ then W is contained in a plane parallel to the YZ plane.

Put $A = \{x : x \in X, \dim[W \cap x \times Y \times Z] \leqq 0\}$. The set $P_x[W]$ is connected and compact. If W is not contained in a plane parallel to the YZ plane then $P_x[W]$ contains more than one point, and hence $P_x[W] = [a, b]$, with $a < b$.

Let x_0 be an interior point of $[a, b]$. Then clearly $W \cap x_0 \times Y \times Z$ disconnects W. Since W is a two-dimensional Cantor manifold we get $\dim[W \cap x_0 \times Y \times Z] \geqq 1$. Hence, $x_0 \notin A$.

Thus we have shown that if W is not contained in a plane parallel to the $Y \times Z$ plane, then there exists an open interval (a, b) in the X axis which is disjoint from A.

For each rational γ, and positive rational ϵ, we denote

$$B_y(\gamma, \epsilon) = \{y : y \in Y, |y - \gamma| < \epsilon\},$$

$$B_z(\gamma, \epsilon) = \{z : z \in Z, |z - \gamma| < \epsilon\},$$

$$S_y(\gamma, \epsilon) = \{x : x \in X, x \times B_y(\gamma, \epsilon) \subset P_{xy}[W]\} \quad \text{and}$$

$$S_z(\gamma, \epsilon) = \{x : x \in X, x \times B_z(\gamma, \epsilon) \subset P_{xz}[W]\}.$$

We shall show now that $X\setminus A \subset \cup S_R(\gamma,\epsilon)$ where the union is taken over all rational γ and $\epsilon > 0$, and $R \in \{y, z\}$. Indeed, let $x_0 \in X\setminus A$, i.e., $\dim[W \cap x_0 \times Y \times Z] \geqq 1$. A set of positive dimension in the plane has positive dimensional projection on at least one of the planes' one dimensional axes. Hence the projection of $W \cap x_0 \times Y \times Z$ on $x_0 \times Y$ or $x_0 \times Z$ contains an interval, and thus a rational interval of type $x_0 \times B_y(\gamma,\varepsilon)$ or $x_0 \times B_z(\gamma,\varepsilon)$. Hence $x_0 \in S_y(\gamma,\varepsilon)$ or $x_0 \in S_z(\gamma,\varepsilon)$. Thus $X\setminus A \subset \cup_{\gamma,\varepsilon} S_R(\gamma,\varepsilon)$ or $A \supset X\setminus \cup_{\gamma,\varepsilon} S_R(\gamma,\varepsilon)$.

We now claim that if $\dim P_{xy}[W] = 1$, then for every γ and ϵ $S_y(\gamma,\epsilon)$ is nowhere dense in X. Indeed, if there exists an interval $I \subset X$ such that $G = S_y(\gamma,\epsilon) \cap I$ is dense in I, then since for each $x \in G$ we have $x \times B_y(\gamma,\epsilon) \subset P_{xy}[W]$, we get $G \times B_y(\gamma,\epsilon) \subset P_{xy}[W]$. But $P_{xy}[W]$ is compact, and therefore $P_{xy}[W] \supset \overline{G \times B_y(\gamma,\epsilon)} \supset \bar{G} \times B_y(\gamma,\epsilon) = I \times B_y(\gamma,\epsilon)$. But $I \times B_y(\gamma,\epsilon)$ is a rectangle. Hence $\dim P_{xy}[W] = 2$.

A similar argument shows that if $\dim P_{xz}[W] = 1$ then for every γ and ϵ $S_z(\gamma,\epsilon)$ is nowhere dense in X. Consequently, if both $P_{xy}[W]$ and $P_{xz}[W]$ are one dimensional, then $\cup_{\gamma,\epsilon} S_R(\gamma,\epsilon)$ is of first category in X. Thus by the Baire category theorem $X\setminus \cup_{\gamma,\epsilon} S_R(\gamma,\epsilon)$ is dense in X. Since $A \supset X\setminus \cup_{\gamma,\epsilon} S_R(\gamma,\epsilon)$, A is dense in X too. But we have seen that if W is not contained in a plane parallel to the $Y \cdot Z$ plane then there exists an interval which is disjoint from A. This proves Lemma 5. □

Let X be an n-dimensional Cantor manifold, and $\psi: X \to R^n$ a function. ψ will be called *dimension preserving* if $\dim \psi[Y] = n$ whenever $\dim Y = n$.

REMARK. By the theorem of Hurewicz [4] mentioned before, zero-dimensional functions are dimension preserving. Let us also remark that the image $\psi[X]$ of an n-dimensional Cantor manifold X under a zero-dimensional function ψ, is again an n-dimensional Cantor manifold. Indeed, if U disconnects $\psi[X]$, then $\psi^{-1}(U)$ disconnects X, and thus $\dim \psi^{-1}(U) \geqq n-2$. By the Hurewicz theorem, we get that $\dim U = \dim \psi(\psi^{-1}(U)) \geqq \dim \psi^{-1}(U) \geqq n-2$.

LEMMA 6. *Let X be a two-dimensional Cantor manifold, and let $\{\varphi_i\}_{i=1}^4$ be four functions in $C(X)$ such that each $f \in C(X)$ admits a representation of the form*

$$f(x) = \sum_{i=1}^{4} g_i(\varphi_i(x)) \quad \textit{with} \quad g_i \in C(R).$$

Then there exists a permutation π of $\{1,2,3,4\}$ and a two-dimensional Cantor *manifold $Y \subset X$ so that the restriction to Y of the four functions* $(\varphi_{\pi(1)}, \varphi_{\pi(2)})$, $(\varphi_{\pi(1)}, \varphi_{\pi(3)})$, $(\varphi_{\pi(2)}, \varphi_{\pi(4)})$, $(\varphi_{\pi(3)}, \varphi_{\pi(4)})$ *(all maps into R^2) are dimension preserving.*

PROOF. We use the notation $(\varphi_i, \varphi_j) = \varphi_{i,j}$. Let us first observe that by Lemma 2, Lemma 4, and the remark following it, each 3-tuple of different functions $\varphi_{i,j,l} = (\varphi_i, \varphi_j, \varphi_l)$ out of the four functions $\{\varphi_i\}_{i=1}^4$, forms a zero-dimensional function $\varphi_{i,j,l}: X \to R^3$. Hence $\varphi_{i,j,l}$ maps two-dimensional Cantor manifolds onto two-dimensional Cantor manifolds.

We shall show now, that given two functions of the type $\varphi_{i,j}$, with a common index i, say $\varphi_{1,2}$, and $\varphi_{1,3}$, they cannot both lower the dimension of the same two-dimensional subset of X. Indeed, assume that both $\varphi_{1,2}$, and $\varphi_{1,3}$ reduce the dimension of a two-dimensional set $U \subset X$. Let Y be a two-dimensional Cantor manifold contained in U. We have $\dim \varphi_{1,2}[Y] = \dim \varphi_{1,3}[Y] = 1$. ($\dim \varphi_{i,j}[Y] = 0$ is clearly impossible).

By our remark, $\varphi_{1,2,3}$ is zero-dimensional, and hence $W = \varphi_{1,2,3}[Y]$ is a two-dimensional Cantor manifold contained in R^3. The two-dimensional projections of W in R^3 are $\varphi_{1,2}[Y]$, $\varphi_{1,3}[Y]$, and $\varphi_{2,3}[Y]$, and by our assumption two of them, $\varphi_{1,2}[Y]$ and $\varphi_{1,3}[Y]$, are one-dimensional. Hence by Lemma 5, W is contained in a plane parallel to the 2, 3 plane. This means that φ_1 is constant on Y. Thus each $f \in C(Y)$ can be represented as $f(x) = \sum_{i=2}^4 g_i(\varphi_i(x))$ contradicting Lemma 4. This proves our assertion.

We come now to the proof of the lemma. If all the six functions $\varphi_{i,j}: X \to R^2$ are dimension preserving, then there is nothing to prove. Assume that one of the functions, say $\varphi_{2,3}$, reduces the dimension of a two-dimensional set $U \subset X$. Let Y be a two-dimensional Cantor manifold contained in U. By our previous assertion, all functions $\varphi_{i,j}$, having a common index with $\varphi_{2,3}$, are dimension preserving on Y, i.e. $\varphi_{1,2}$, $\varphi_{1,3}$, $\varphi_{2,4}$, $\varphi_{3,4}$ are dimension preserving on Y. This proves Lemma 6. □

PROOF OF THEOREM 2. Theorem 2 will be proved by showing that if X is a two dimensional space and $\{\varphi_i\}_{i=1}^4$ are four functions in $C(X)$ such that each $f \in C(X)$ admits a representation

$$f(x) = \sum_{i=1}^{4} g_i(\varphi_i(x)) \quad \text{with} \quad g_i \in C(R), \quad \text{then}$$

Lemma 3 is contradicted, i.e., given $M > 0$, there exists an $f \in C(X)$, $\|f\| = 1$, such that in every representation of f in the above form, there exists an index i such that $\|g_i\| \geqq M$.

As usual, we assume that X is a two-dimensional Cantor manifold, and by Lemma 6, we may assume that the functions $\varphi_{1,2}$, $\varphi_{1,3}$, $\varphi_{2,4}$, $\varphi_{3,4}$ are dimension preserving on X. We use the letters $\alpha, \beta, \gamma, \delta$ to denote the images of points $x \in X$ under φ_i $i = 1,2,3,4$ respectively. As in the proof of Lemma 4, we call a sequence $\{x_i\}_{i=1}^k \subset X$ an alternating sequence if we have $\varphi_2(x_i) = \varphi_2(x_{i+1})$ for i odd, and $\varphi_3(x_i) = \varphi_3(x_{i+1})$ for i even. A pair of m-tuples of points $\{x_i\}_{i=1}^m$ $\{x_i\}_{i=m+1}^{2m}$ will be called a doubly-alternating sequence relative to α (resp. relative to δ) if both $\{x_i\}_{i=1}^m$ and $\{x_i\}_{i=m+1}^{2m}$ are alternating sequences, and in addition $\varphi_1(x_i) = \varphi_1(x_{m+i}) = \alpha_i$, $i = 1,2,\cdots,m$, (respectively $\varphi_4(x_i) = \varphi_4(x_{m+1}) = \delta_i$, $i = 1,2,\cdots,m$.)

Let G and G' be subsets of X and let k be an integer. We use the notation $G' <_{1,k} G$ if $G' \subset G$, and for every set of k points $\{\alpha_i\}_{i=1}^k \subset \varphi_1[G']$, so that α_i's with even indices are different from α_i's with odd indices, there exists an alternating sequence $\{x_i\}_{i=1}^k$ in G with $\varphi_1(x_i) = \alpha_i$, $1 \leqq i \leqq k$. The relation $G' <_{4,k} G$ is defined similarly with $\{\delta_i\}_{i=1}^k$ and φ_4 replacing $\{\alpha_i\}_{i=1}^k$ and φ_1 respectively.

Since the functions $\varphi_{1,2}, \varphi_{1,3}, \varphi_{2,4}, \varphi_{3,4}$ are dimension preserving, we can show, using the same method as in the proof of Lemma 4, that given any $G \subset X$ with non-empty interior, and any integer $k > 0$, there exist subsets G', G'' of X, with non-empty interior so that $G' <_{1,k} G$, and $G'' <_{4,k} G$. (This is done by constructing $G = G_k \supset_{1,2} G_{k-1} \supset_{1,3} G_{k-2} \supset \cdots \supset_{1,2} G_1 = G'$). Using the methods of the proof of Lemma 4, we are also able to show that each subset G of X with non-empty interior contains doubly-alternating sequences of arbitrary length. (Relative to α or δ).

Let $k > 0$ be any integer. The set $\varphi_{1,2}[X]$ contains a rectangle

$$S = \{(\alpha\beta) : \alpha' \leqq \alpha \leqq \alpha''; \ \beta' \leqq \beta \leqq \beta''\}.$$

Choose β^* so that $\beta'' < \beta^* < \beta''$, and set:

$$S_1 = \{(\alpha\beta) : \alpha' \leqq \alpha \leqq \alpha'', \ \beta' \leqq \beta \leqq \beta^*\}$$

$$S_2 = \{(\alpha\beta) : \alpha' \leqq \alpha \leqq \alpha'', \ \beta^* \leqq \beta \leqq \beta''\}$$

(see Fig. 4).

Set $G_k = \varphi_{1,2}^{-1}(S_1)$. The set G_k is closed with non-empty interior, and according to our comments, a subset G'_k of G_k can be constructed, so that G'_k is closed with non-empty interior, and $G'_k <_{1,2k} G_k$. $\varphi_{1,2}[G'_k]$ contains a non-empty open plane set. Thus, $\varphi_1[G'_k]$ contains an interval $[\alpha^*, \alpha^{**}]$ which is clearly contained in $[\alpha', \alpha'']$. (see Fig. 4). Denote:

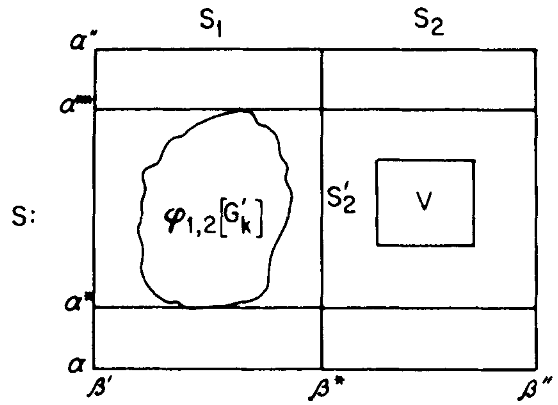

Fig. 4

$$S_2' = \{(\alpha\beta) : \alpha^* < \alpha < \alpha^{**},\ \beta^* < \beta < \beta''\},$$

and let V be a closed rectangle with sides parallel to the axes, contained in S_2'. Set $X' = \varphi_{1,2}^{-1}(V)$. Obviously we have:

(1) $$X' \cap G_k = \phi$$

(2) $$\varphi_1[X'] \subset [\alpha^*, \alpha^{**}] \subset \varphi_1[G_k'].$$

The space X' is a compact two-dimensional metric space. Operating in a similar manner on X' (or on a Cantor manifold contained in it), we construct in X' closed sets with non-empty interior H_k, H_k' and X'' so that:

(3) $$H_k' \underset{4,2k}{<} H_k$$

(4) $$\varphi_4[X''] \subset \varphi_4[H_k']$$

(5) $$X'' \cap H_k = \phi.$$

Again in X'' we construct closed sets with non-empty interior G_{k-1}, G_{k-1}', and X''' so that:

(6) $$G_{k-1}' \underset{1,2k}{<} G_{k-1}$$

(7) $$\varphi_1[X'''] \subset \varphi_1[G_{k-1}']$$

(8) $$X''' \cap G_{k-1} = \phi.$$

Continuing in this manner we construct sets $\{G_i\}_{i=1}^k$, $\{G_i'\}_{i=1}^k$, $\{H_i\}_{i=1}^k$, $\{H_i'\}_{i=1}^k$ so that:

(i) $G'_i <_{1,2k} G_i$, $H'_i <_{4,2k} H_i$ for $1 \leqq i \leqq k$.

(ii) $\varphi_1[H_i] \subset \varphi_1[G'_i]$ for $1 \leqq i \leqq k$ and $\varphi_4[G_{i-1}] \subset \varphi_4[H'_i]$ for $2 \leqq i \leqq k$.

(iii) The sets $\{H_i\}_{i=1}^k$ and $\{G_i\}_{i=1}^k$ are all mutually disjoint. The next steps in the proof are, unfortunately, somewhat involved. (By that we don't mean that the previous were simple....) To fix the ideas, let us consider the following example where $k = 2$.

In this case we have:

$$G'_1 \underset{1,4}{<} G_1, \quad G'_2 \underset{1,4}{<} G_2, \quad H'_1 \underset{4,4}{<} H_1, \quad H'_2 \underset{4,4}{<} H_2.$$

$$\varphi_1[H_1] \subset \varphi_1[G'_1]; \quad \varphi_1[H_2] \subset \varphi_1[G'_2] \quad \varphi_4[G_1] \subset \varphi_4[H'_2],$$

and the sets G_1, G_2, H_1, H_2 are mutually disjoint.

Let $\delta'_1, \delta'_2, \delta'_3, \delta'_4$ be distinct points in $\varphi_4[H'_1]$. Since $H'_1 <_{4,4} H_1$, there exists an alternating sequence $\{x_i^1\}_{i=1}^4$ in H_1, with $\varphi_4(x_i^1) = \delta_i^1$. Thus, the $\{x_i^1\}_{i=1}^4$ are distinct. Set $\varphi_1(x_i^1) = \alpha_i^1$. $i = 1, 2, 3, 4$. We have: $\{\alpha_i^1\}_{i=1}^4 \subset \varphi_1[H_1] \subset \varphi_1[G'_1]$, and $G'_1 <_{1,4} G_1$. Hence if $\{\alpha_1^1, \alpha_3^1\} \cap \{\alpha_2^1, \alpha_4^1\} = \phi$, then there exist in G_1 an alternating sequence $\{x_i^2\}_{i=1}^4$, with $\varphi_1(x_i^2) = \alpha_i^1$, and $\{x_1^2, x_3^2\} \cap \{x_2^2, x_4^2\} = \phi$. Set $\varphi_4(x_i^2) = \delta_i^2$. Then $\{\delta_i^2\}_{i=1}^4 \subset \varphi_4[G_1] \subset \varphi_4[H'_2]$. Hence if $\{\delta_1^2, \delta_3^2\} \cap \{\delta_2^2, \delta_4^2\} = \phi$, there exists in H_2 an alternating sequence $\{x_i^4\}_{i=1}^4$ with $\varphi_4(x_i^3) = \delta_i^2$, and $\{x_1^3, x_3^3\} \cap \{x_2^3, x_4^3\} = \phi$. Setting $\varphi_1(x_i^3) = \alpha_i^2$, and hoping that $\{\alpha_1^2, \alpha_3^2\} \cap \{\alpha_2^2, \alpha_4^2\} = \phi$, we can find in G_2 an alternating sequence $\{x_i^4\}_{i=1}^4$ with $\varphi_1(x_i^4) = \alpha_i^2$.

In this manner, the 16 points $\{x_i^j\}$ $1 \leqq i, j \leqq 4$ are constructed, with

$$\{x_i^j : i + j \equiv 1 \pmod 2\} \cap \{x_i^j : i + j \equiv 0 \pmod 2\} = \phi,$$

and:

TABLE II

$\varphi(x_1^1) = \alpha_1^1, \beta_1^1, \gamma_1^1, \delta_1^1$	$\varphi(x_1^2) = \alpha_1^1, \beta_1^2, \gamma_1^2, \delta_1^2$
$\varphi(x_2^1) = \alpha_2^1, \beta_1^1, \gamma_2^1, \delta_2^1$	$\varphi(x_2^2) = \alpha_2^1, \beta_1^2, \gamma_2^2, \delta_2^2$
$\varphi(x_3^1) = \alpha_3^1, \beta_2^1, \gamma_2^1, \delta_3^1$	$\varphi(x_3^2) = \alpha_3^1, \beta_2^2, \gamma_2^2, \delta_3^2$
$\varphi(x_4^1) = \alpha_4^1, \beta_2^1, \gamma_3^1, \delta_4^1$	$\varphi(x_4^2) = \alpha_4^1, \beta_2^2, \gamma_3^2, \delta_4^2$
$\varphi(x_1^3) = \alpha_1^2, \beta_1^3, \gamma_1^3, \delta_1^2$	$\varphi(x_1^4) = \alpha_1^2, \beta_1^4, \gamma_1^4, \delta_1^3$
$\varphi(x_2^3) = \alpha_2^2, \beta_1^3, \gamma_2^3, \delta_2^2$	$\varphi(x_2^4) = \alpha_2^2, \beta_1^4, \gamma_2^4, \delta_2^3$
$\varphi(x_3^3) = \alpha_3^2, \beta_2^3, \gamma_2^3, \delta_3^2$	$\varphi(x_3^4) = \alpha_3^2, \beta_2^4, \gamma_2^4, \delta_3^3$
$\varphi(x_4^3) = \alpha_4^2, \beta_2^3, \gamma_3^3, \delta_4^2$	$\varphi(x_4^4) = \alpha_4^2, \beta_2^4, \gamma_3^4, \delta_4^3$

If $f \in C(X)$ admits the representation $f(x) = \Sigma_{i=1}^{4} g_i(\varphi_i(x))$, then following Table II we get:

$$W = \sum_{j=1}^{4} \sum_{i=1}^{4} (-1)^{i+j} f(x_i^j) = g_3(\gamma_1^1) + g_3(\gamma_4^1) + g_3(\gamma_1^2) + g_3(\gamma_4^2)$$

$$+ g_3(\gamma_1^3) + g_3(\gamma_4^3) + g_3(\gamma_1^4) + g_3(\gamma_4^4) + \sum_{i=1}^{4} g_4(\delta_i^3),$$

i.e. $3 \cdot 4 = 3 \cdot 2k$ summands. Since the sets $N = \{x_i^j : i + j \equiv 1 \pmod 2\}$ and $L = \{x_i^j : i + j \equiv 0 \pmod 2\}$ are disjoint in X, there exists in $C(X)$ an f with $\|f\| = 1$, $f(N) = -1$, and $f(L) = +1$. For this f we get:

$$W = \sum_{j=1}^{4} \sum_{i=1}^{4} (-1)^{i+j} f(x_i^j) = 4^2,$$

(i.e. $W = (2k)^2 = 4k^2$). Hence, for g_3 or g_4, we get $\|g\| \geqq (4k^2/3 \cdot 2k) = 2k/3$. One observes easily that the last inequality holds if a similar construction is carried out with an arbitrary k. The following is a precise realization of the above ideas.

Let $\{\delta_i^1\}_{i=1}^{2k}$ be distinct points in $\varphi_4[H_1']$. Since $H_1' <_{4,2k} H_1$, there exists an alternating sequence $\{x_i^1\}_{i=1}^{2k}$ in H_1, with $\varphi_4(x_i^1) = \delta_i^1$. Thus, the $\{x_i^1\}_{i=1}^{2k}$ are distinct. If $f \in C(X)$ admits the representation:

$$f(x) = \sum_{i=1}^{4} g_i(\varphi_i(x)),$$

then it follows from the definition of alternating sequence that:

$$\text{(A)} \qquad \sum_{i=1}^{2k} (-1)^{i+1} f(x_i^1) = \sum_{i=1}^{2k} (-1)^{i+1} g_4(\delta_i^1) + \sum_{i=1}^{2k} (-1)^{i+1} g_1(\alpha_i^1)$$

$$+ g_3(\gamma_1) - g_3(\gamma_{k+1}),$$

where $\alpha_i^1 = \varphi_1(x_i^1)$ $1 \leqq i \leqq 2k$, $\gamma_1 = \varphi_3(x_1^1)$, and $\gamma_{k+1} = \varphi_3(x_{2k}^1)$.

It may happen that an α_i^1 with an even index i will be equal to an α_j^1 with an odd index j. Suppose we have m such pairs (i, j) i even, j odd and $\alpha_i^1 = \alpha_j^1$. $(m \leqq k)$. Thus the sum $\Sigma_{i=1}^{2k}(-1)^{i+1} g_i(\alpha_i^1)$ reduces to a sum with $2(k - m)$ summands at most. Let us reorder the $[2(k-m)]\,\alpha_i^1$'s which appear in the reduced sum by $\{\tilde{\alpha}_j^1\}_{j=1}^{2(k-m)}$ in such a manner that an $\tilde{\alpha}_j^1$ gets an odd index j if and only if $\tilde{\alpha}_j^1 = \alpha_i^1$ with i odd. Such a reordering is possible since pairs of elements, one with odd index and the other with even index were removed from $\{\alpha_i^1\}_{i=1}^{2k}$. Clearly, in the set $\{\tilde{\alpha}_j^1\}_{j=1}^{2(k-m)}$ elements with an odd index are different from elements with an even index, and:

(B) $$\sum_{i=1}^{2k}(-1)^{i+1}g_1(\alpha_i^1)=\sum_{i=1}^{2(k-m)}(-1)^{i+1}g_1(\tilde{\alpha}_i^1).$$

We have $\{\tilde{\alpha}_i^1\}_{i=1}^{2(k-m)}\subset\varphi_1[H_1]\subset\varphi_1[G_1']$, and $G_1'<_{1,2k}G_1$. Thus, there exists an alternating sequence $\{x_i^2\}_{i=1}^{2(k-m)}$ in G_1, with $\varphi_1(x_i^2)=\tilde{\alpha}_i^1$ $1\leqq i\leqq 2(k-m)$, and $\{x_i^2\}_{i=1}^{2(k-m)}\cap\{x_i^1\}_{i=1}^{2k}=\phi$ since $H_1\cap G_1=\phi$. Clearly the points x_i^2 with an even index i differ from the points x_i^2 with an odd index i since the corresponding $(\tilde{\alpha}_i^1)$'s differ. It follows from the properties of an alternating sequence that:

(C) $$\sum_{i=1}^{2(k-m)}(-1)^i f(x_i^2)=\sum_{i=1}^{2(k-m)}(-1)^i g_1(\tilde{\alpha}_i^1)+\sum_{i=1}^{2(k-m)}(-1)^i g_4(\delta_i^2)+\left\{\begin{matrix}\text{2 summands}\\ \text{of the form}\\ g_3(\gamma)\end{matrix}\right\}$$

$$=-\sum_{i=1}^{2(k-m)}(-1)^{i+1}g_1(\tilde{\alpha}_i^1)+\sum_{i=1}^{2(k-m)}(-1)^i g_4(\delta_i^2)+\left\{\begin{matrix}\text{2 summands}\\ \text{of the form}\\ g_3(\gamma)\end{matrix}\right\}$$

where $\delta_i^2=\varphi_4(x_i^2)$ $1\leqq i\leqq 2(k-m)$.

Thus, if we sum (A) and (C) we get in view of (B):

(D) $$\sum_{i=1}^{2k}(-1)^{i+1}f(x_i^1)+\sum_{i=1}^{2(k-m)}(-1)^i f(x_i^2)=\sum_{i=1}^{2k}(-1)^{i+1}g_4(\delta_i^1)+\sum_{i=1}^{2(k-m)}(-1)^i g_4(\delta_i^2)+\{\text{4 summands of the form } g_3(\gamma)\}.$$

Let us select, in $G_1\setminus\{x_i^2\}_{i=1}^{2(k-m)}$, a doubly alternating sequence relative to α $\{x_i^2\}_{i=2(k-m)+1}^{2k-m}$, $\{x_i^2\}_{i=2k-m+1}^{2k}$ of distinct points. Set $\delta_i^2=\varphi_4(x_i^2)$, $2(k-m)+1\leqq i\leqq 2k$. Then we get (see Table I in the proof of Lemma 4):

(E) $$\sum_{i=2(k-m)+1}^{2k}(-1)^i f(x_i^2)=\sum_{i=2(k-m)+1}^{2k}(-1)^i g_4(\delta_i^2)+\left\{\begin{matrix}\text{4 summands}\\ \text{of the form}\\ g_2(\beta)\text{ or }g_3(\gamma)\end{matrix}\right\}$$

(a summand of the form $g_2(\beta)$ may appear in (E) if m is odd). By summing (D) and (E) we get:

(F) $$\sum_{j=1}^{2}\sum_{i=1}^{2k}(-1)^{i+j}f(x_i^j)=\sum_{i=1}^{2k}(-1)^{i+1}g_4(\delta_i^1)+\sum_{i=1}^{2k}(-1)^i g_4(\delta_i^2)+\{\text{8 summands of the form } g_2(\beta)\text{ or }g_3(\gamma)\}.$$

The points $\{\delta_i^2\}_{i=1}^{2k}$ are all in $\varphi_4[G_1]$, and $\varphi_4[G_1]\subset\varphi_4[H_2']$. Thus, since $H_2'<_{4,2k}H_2$, we can repeat the process carried out with the $\{\alpha_i^1\}_{i=1}^{2k}$, i.e., remove the m' pairs (δ_i^2, δ_j^2) with $\delta_i^2 = \delta_j^2\, i$ even and j odd, reorder the remaining δ_i^2's in the prescribed manner as $\{\tilde{\delta}_i^2\}_{i=1}^{2(k-m')}$, select in H_2 an alternating sequence $\{x_i^3\}_{i=1}^{2(k-m')}$ with $\varphi_4(x_i^3) = \tilde{\delta}_i^2$, and then select in $H_2 \setminus \{x_i^3\}_{i=1}^{2(k-m')}$ a double alternating sequence $\{x_i^3\}_{i=2(k-m')+1}^{2k-m'}$, $\{x_i^3\}_{i=2k-m'+1}^{2k}$ relative to δ. By (F) we then get:

(G) $$\sum_{j=1}^{3}\sum_{i=1}^{2k}(-1)^{i+j}f(x_i^j) = \sum_{i=1}^{2k}(-1)^{i+1}g_4(\delta_i^1) + \sum_{i=1}^{2k}(-1)^{i+1}g_1(\alpha_i^2)$$

$$+\{\text{less than } 3\cdot 6 \text{ summands of the form } g_2(\beta) \text{ or } g_3(\gamma)\},$$

where $\alpha_i^2 = \varphi_1(x_i^3)$.

REMARK. We use "less than $3\cdot 6$ summands" in the brackets, since each j adds 6 such summands (2 by the alternating sequence, and 4 by the double alternating sequence) except $j = 1$ where there is no alternating sequence, and just 2 summands of the form $g_3(\gamma)$ are added.

Repeating this process inductively $2k$ times, we get:

(H) $$\sum_{j=1}^{2k}\sum_{i=1}^{2k}(-1)^{i+j}f(x_i^j) = \sum_{i=1}^{2k}(-1)^{i+1}g_4(\delta_i^1) + \sum_{i=1}^{2k}(-1)^{i}g_4(\delta_i^k)$$

$$+\{\text{less than } 2k\cdot 6 \text{ summands of the form } g_2(\beta) \text{ or } g_3(\gamma)\},$$

where $\delta_i^k = \varphi_4(x_i^{2k})$ $1 \leqq i \leqq 2k$.

The points $\{x_i^j\}$ $1 \leqq i,\ j \leqq 2k$ were selected in such a manner that $j_1 \neq j_2$ implies $\{x_i^{j_1}\}_{i=1}^{2k} \cap \{x_i^{j_2}\}_{i=1}^{2k} = \phi$, and such that for fixed j_0 $x_i^{j_0}$ with odd i differs from all the $x_i^{j_0}$'s with even i. Thus the sets

$$N = \{x_i^j : i + j \equiv 1 \pmod 2\} \text{ and}$$

$$L = \{x_i^j : i + j \equiv 0 \pmod 2\}$$

are disjoint.

Let $f \in C(X)$ be such that $\|f\| = 1$, $f[L] = 1$ and $f[N] = -1$. Suppose f admits the representation:

$$f(x) = \sum_{i=1}^{4} g_i(\varphi_i(x)).$$

Then by (H) we get

(I) $$4k^2 = \sum_{x\in L} f(x) - \sum_{x \in N} f(x) = \sum_{j=1}^{2k}\sum_{i=1}^{2k}(-1)^{i+j}f(x_i^j) =$$

$= \{\text{less than } (2k + 2k + 2k \cdot 6) \text{ summands of the form } g_i(t)\ i = 1,2,3,4\}.$

Thus $\max_{1 \le i \le 4} \|g_i\| \ge 4k^2/16k = k/4$.

Since k was arbitrary this proves Theorem 2. □

3. The case $\dim X = 1$

We conclude with a special result concerning the case $n = 1$.

THEOREM 3. *Let T denote the circle. Let ψ_1, ψ_2 be any two functions in $C(T)$. Then there exists an $f \in C(T)$ such that f cannot be represented in the form $f(x) = g_1(\psi_1(x)) + g_2(\psi_2(x))$, $g_i \in C(R)$.*

Let X be a space with $\dim X \le 1$. Clearly, each $f \in C(X)$ can be represented in our usual form using one $\varphi \in C(X)$ (i.e. $f(x) = g(\varphi(x))$ if and only if X is topologically contained in an interval. If two functions φ_1, φ_2 can do the job (i.e. each $f \in C(X)$ admits a representation $f(x) = g_1(\varphi_1(x)) + g_2(\varphi(x))$, then by Theorem 3, X does not contain a circle. Then, if X is assumed to be connected, and locally connected, it is a tree (dendrite). It is known that if X is a finite tree with branching index not greater than 3, then two functions φ_1, φ_2 are enough. In the case of infinite trees, it is not known whether two functions are enough or not, however, three are clearly enough by Theorem 1*. For a detailed treatment of the case where X is a tree, see [1].

PROOF OF THEOREM 3. Let $\psi_1, \psi_2 \in C(T)$.

Let V be a subset of T. Set

$$V^1 = \{t : t \in V,\ \mathrm{card}[\psi_1^{-1}(\psi_1(t) \cap V] \ge 2\}$$

$$V^2 = \{t : t \in V,\ \mathrm{card}[\psi_2^{-1}(\psi_2(t)) \cap V] \ge 2\}$$

(card $[U]$ denotes the cardinality of the set U),

and

$$V_{(1)} = V^1 \cap V^2.$$

Consider the sequence $\{T_n\}_{n=0}^{\infty}$ of subsets of T defined inductively by:

$$T_0 = T; \quad T_{n+1} = (T_n)_{(1)}.$$

We claim that $\bigcap_{n=0}^{\infty} T_n \ne \phi$. We shall prove this by showing that for each n, $T \setminus T_n$ is finite. $T \setminus T_0$ is empty by definition. Assume that

$$T \setminus T_n = \{t_1 \cdots t_k\}.$$

$$(T_n)^1 = \{t : t \in T_n,\ \mathrm{card}[\psi_1^{-1}(\psi_1(t)) \cap T_n] \ge 2\}.$$

Let $t \in T_n \setminus (T_n)^1$. Then clearly card$[\psi_1^{-1}(\psi_1(t))] \leqq k+1$, and hence $\psi_1^{-1}(\psi_1(t)) = \{t, \gamma_1, \gamma_2, \cdots \gamma_m\}$ $1 \leqq m \leqq k$ with $\{\gamma_1 \cdots \gamma_m\} \subset \{t_1 \cdots t_k\}$. There are at most two points x in T with card$[\psi_1^{-1}(\psi_1(x))] = 1$. Observe also that if $t, t' \in T_n \setminus (T_n)^1$, then $\psi_1^{-1}(\psi_1(t)) \cap \psi_1^{-1}(\psi_1(t')) = \phi$. For otherwise, $\psi_1(t) = \psi_1(t')$ which implies $t, t' \in (T_n)^1$. Consequently, there are at most $k+2$ points in $T_n \setminus (T_n)^1$, thus $T \setminus (T_n)^1$ is finite.

The same argument shows that $T \setminus (T_n)^2$ is finite. Hence $T \setminus T_{n+1} = T \setminus ((T_n)^1 \cap (T_n)^2)$ is finite.

REMARK. Since card$[T \setminus T_1] \leqq 4$, one can easily obtain card$[T \setminus T_n] \leqq 4 \cdot 3^{n-1}$. Let n be an integer, and let $t_1 \in T_n$. Hence, there exists a $t_2 \in T_{n-1}$, $t_2 \neq t_1$, with $\psi_1(t_1) = \psi_1(t_2) = \alpha_1$. Set $\psi_2(t_1) = \beta_1$. There exists a $t_3 \in T_{n-2}$, $t_3 \neq t_2$ so that $\psi_2(t_2) = \psi_2(t_3) = \beta_2$. There exists $t_4 \in T_{n-3}$, $t_3 \neq t_4$ so that $\psi_1(t_3) = \psi_1(t_4) = \alpha_2$. Continuing inductively, we construct a sequence $\{t_1 \cdots t_n\}$ with:

$$(*): \qquad \psi_1(t_{2k-1}) = \psi_1(t_{2k}), \quad \psi_2(t_{2k}) = \psi_2(t_{2k+1}) \quad (k \geqq 1).$$

If for each n the sequence $\{t_1 \cdots t_n\}$ consists of distinct elements, then for each n there exists $f_n \in C(T)$, so that $\|f_n\| = 1$, $f_n(t_{2k}) = 1, f_n(t_{2k-1}) = -1$ $(k \geqq 1)$. Suppose $f_n(t) = g_1(\psi_1(t)) + g_2(\psi_2(t))$. Then by $(*)$ (for n even), $n = \Sigma_{i=1}^n (-1)^{i+1} f(t_i) = g_2(\beta_1) + g_2(\beta_{n/2})$. Thus $\|g_2\| \geqq n/2$. Since n was arbitrary, Lemma 3 implies Th. 3.

If for some n, there exist in the sequence $\{t_1 \cdots t_n\}$ two points $t_k = t_m$ with $1 \leqq k < m \leqq n$, then, $(*)$ implies that:

$$(+) \qquad \sum_{i=k}^{m} (-1)^i f(t_i) = 0, \quad \text{or} \quad \sum_{i=k}^{m-1} (-1)^i f(t_i) = 0$$

for each f which can be represented by $f(t) = g_1(\psi_1(t)) + g_2(\psi_2(t))$. But since there are $f \in C(T)$ which do not satisfy $(+)$, the theorem is proved also in this case. □

Note added in proof. We can show that the theorem is valid also for $n = 3$, i.e. that for no X with $\dim X = 3$ will 6 functions $\{\varphi_i\}_{i=1}^6$ suffice in Theorem 1*. For $n > 3$ we have till now only partial results.

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